A CONDITION OF UNIFORM EXPONENTIAL STABILITY FOR SEMIGROUPS

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Abstract. The aim of this paper is to prove that the uniform exponential stability of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ (acting on a complex Hilbert space H) can be derived as a consequence of the well behavior of its numerical range in a suitable Orlicz space. More precisely, assuming that there exists an Orlicz space $E = (L^{\Phi}, \rho^{\Phi})$ over $\mathbf{R}_{+}$ such that

$$\liminf_{\alpha \downarrow 0} [\alpha || \exp_{-\alpha} ||_{E^{\star}}] = 0$$

and

$$\sup_{||x|| \leq 1} \rho^{\Phi}(|\langle T(\cdot)x, x\rangle|) \leq M < \infty,$$

then the uniform growth bound ω_0 of the semigroup verifies an estimate of the form

$$\omega_0 \leq M_{\beta} := \beta - (2M|| \exp_{-\beta} ||_{E^{\star}})^{-1} < 0$$

for some positive number β. As an application, the well posedness of an abstract infinite time Cauchy problem is discussed.

1. Introduction

Let H be a complex Hilbert space and let $1 \leq p < \infty$. Recall that a semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ on H is called:

- *weakly-L^p-stable* if for every $x, y \in H$ we have

$$\int_0^{\infty} |\langle T(t)x, y\rangle|^p dt < \infty;$$

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- *uniformly exponentially stable* if its uniform growth bound is negative, that is
$$\omega_0(\mathbf{T}) := \lim_{t\to\infty} \frac{\ln ||T(t)||}{t} < 0,$$
or, equivalently, if
$$||T(t)|| \leq Ne^{-\nu t} \text{ for all } t \geq 0.$$
for some positive constants N and ν.

It is clear that each uniformly exponentially semigroup is weakly-L^p-stable. In 1983 A. J. Pritchard and J. Zabczyk [9] raised the problem whether every weakly-L^p-stable semigroup is uniformly exponentially stable. The answer is positive and a solution can be found in [3], [11]. In this note we extend their result to the more general framework of Orlicz spaces. In order to formulate our generalization we shall need a preparation on Orlicz spaces. For further details the reader is referred to [4], [5], [6], [1] and references therein.

The Orlicz spaces over $\mathbb{R}_+$ are attached to nondecreasing convex functions $\Phi : [0,\infty) \to [0,\infty]$ such that $\Phi(0) = \Phi(0+) = 0$ and Φ is not identically 0 or ∞ on $(0,\infty)$. We denote by L^Φ the set of all complex-valued measurable functions f defined on $\mathbb{R}_+$ for which there exists a positive λ such that $\int_0^\infty \Phi(\lambda|f(t)|)dt < \infty$. Clearly, L^Φ is a linear space with respect to the usual operations and we can turn L^Φ into an Orlicz space by considering on it the norm ρ^Φ, where
$$\rho^\Phi(f) := \inf\{k > 0 : \int_0^\infty \Phi(k^{-1}|f(t)|)dt \leq 1\}.$$

If Φ satisfies the Δ_2-condition i.e., there exists a positive constant C such that
$$\Phi(2t) \leq C\Phi(t) \text{ for all } t \geq 0,$$
then the dual space $\left(L^\Phi\right)^\star$ is also an Orlicz space. Moreover, in this case $\left(L^\Phi\right)^\star$ can be identified with $L^{\Phi^\star}$, where
$$\Phi^\star(t) := \sup_{s\geq 0}(ts - \Phi(s)), \quad t \geq 0$$
is the Legendre transform of Φ.

Clearly, all Lebesgue spaces $L^p(\mathbb{R}_+)$ (for $1 \leq p < \infty$) are examples of Orlicz spaces which satisfy the Δ_2-condition.

We can now state our main result:

Theorem 1. *Let* $\mathbf{T} = \{T(t)\}_{t\geq 0}$ *be a strongly continuous semigroup acting on a complex Hilbert space* H. *Then* $\mathbf{T}$ *is uniformly exponentially* *stable if (and only if) it verifies the following condition*

$$M = \sup_{||x|| \leq 1} \rho^{\Phi}(|\langle T(\cdot)x, x\rangle|) < \infty, \tag{1.1}$$

with respect to an Orlicz space $E = \left(L^{\Phi}, \rho^{\Phi}\right)$ *whose dual space* $E^{\star}$ *has the property that*

$$\liminf_{\alpha \downarrow 0} [\alpha || \exp_{-\alpha} ||_{E^{\star}}] = 0. \tag{1.2}$$

The necessity of the condition (1.1) is straightforward. In fact, if the semigroup $\mathbf{T}$ is uniformly exponentially stable, then (1.1) works for all Orlicz spaces. The sufficiency part is detailed in the next section.

As shows the case where $\mathbf{T}$ is the left translation semigroup on $H = L^2(\mathbb{R})$ and $E = L^{\infty}(\mathbb{R}_+)$, the condition (1.2) is essential for the validity of Theorem 1.

In the special case where $\Phi(t) = t^p$ (for $1 \leq p < \infty$), the result of Theorem 1 was first proved by G. Weiss [11]. Clearly, in that case the condition (1.2) is automatically fulfilled. Our result covers more general Orlicz functions Φ which satisfy the Δ_2-condition and $\lim_{t \to 0+} t\rho^{\Phi^*}(\exp_{-t}) = 0$ such as $\Phi(t) = e^t - t - 1$. In this case

$$\Phi^*(t) = (t+1)\ln(t+1) - t$$

and

$$\begin{aligned} \rho^{\Phi^*}(\exp_{-\alpha}) &= \inf\{k > 0 : \int_0^{\infty} \Phi^*(\frac{e^{-\alpha t}}{k})dt \leq 1\} \\ &= \inf\{k > 0 : \frac{1}{\alpha}\int_0^{1/k} \frac{u+1}{u}\ln(u+1)du - \frac{1}{k\alpha} \leq 1\} \\ &= \sup\{b > 0 : \int_0^{b} \frac{u+1}{u}\ln(u+1)du \leq b + \alpha\} = b_0, \end{aligned}$$

where b_0 is the unique solution of the following equation (in variable x),

$$\int_0^{x} \frac{u+1}{u}\ln(u+1)du = x + \alpha.$$

The map $\alpha : x \to \int_0^x \frac{u+1}{u}\ln(u+1)du - x$ (from $[0, \infty)$ into $[0, \infty)$) is surjective and also increasing, so that its inverse is continuous. Consequently α^{-1} is bounded on $[0, 1]$, which yields $\lim_{t \to 0+} t\rho^{\Phi^*}(\exp_{-t}) = 0$.

J.M.A.M. van Neerven [7] has noticed that any bounded strongly continuous semigroup (acting on a complex Hilbert space H) is uniformly exponentially stable if there exists a nondecreasing function

$\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(t) > 0$ for $t > 0$ and

$$\int_0^\infty \varphi(|\langle T(t)x, y\rangle|)dt < \infty, \quad \text{for all } x, y \in H.$$

We leave open the question whether the boundedness condition can be dropped.

2. Proof of Theorem 1

Proof. We already noticed that only the sufficiency part needs an argument. For this, we need the remark that the condition of boundedness (1.1) yields

$$N = \sup_{||x||,\|y\|\le 1} \rho^\Phi(|\langle T(\cdot)x, y\rangle|) \le 2M < \infty,$$

as a consequence of the polarization identity

$$\langle T(t)x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \langle T(t)(x + i^k y), x + i^k y\rangle.$$

The next step is to motivate the existence of the improper integral

$$\int_0^\infty u^\star(t)T(t)xdt := \lim_{s\to\infty}\int_0^s u^\star(t)T(t)xdt,$$

for all $u^\star \in E^\star$ and $x \in H$. In terms of series, this limit means the convergence of

$$\sum_{n=0}^{\infty}\int_{s_n}^{s_{n+1}} u^\star(t)T(t)xdt \tag{2.1}$$

for all positive sequences $(s_n)_n$, with $s_0 = 0$, which are increasing to ∞. This can be derived from a classical result due to Orlicz-Pettis, which asserts that every weakly unconditionally convergent series (in a Banach space) is also unconditionally convergent. In fact,

$$\begin{aligned}
\sum_{n=0}^{N}\left|\langle \int_{s_n}^{s_{n+1}} u^\star(t)T(t)xdt, y\rangle\right| &= \sum_{n=0}^{N} e^{i\lambda_n}\langle \int_{s_n}^{s_{n+1}} u^\star(t)T(t)xdt, y\rangle \\
&= \langle \sum_{n=0}^{N}\int_{s_n}^{s_{n+1}} e^{i\lambda_n}u^\star(t)T(t)xdt, y\rangle \\
&= \langle \int_0^{s_{N+1}}\left(\sum_{n=0}^{N} e^{i\lambda_n}\chi_{[s_n,s_{n+1})}(t)\right)u^\star(t)T(t)xdt, y\rangle \\
&\le M||u^\star||_{E^\star}||x||\,\|y\|,
\end{aligned}$$

for all $x, y \in H$ and $N \in \mathbb{N}$, which yields the weak unconditional convergence of the series (2.1).

Since

$$\left\|\int_0^s u^\star(t)T(t)xdt\right\| = \sup_{||y||\le 1}\left|\langle\int_0^s u^\star(t)T(t)xdt, y\rangle\right|$$

we get also the inequality

$$\left\|\int_0^\infty u^\star(t)T(t)xdt\right\| \le M||x||||u^\star||_{E^\star}.$$

As well known, the dual space of any Orlicz space is a rearrangement invariant Banach function space which contains the space $L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$. See [1], [5], [6]. Thus for each $\beta > 0$ the function $\exp_{-\beta}$ belongs to $E^\star$. Moreover, if $\lambda \in \mathbb{C}$ and $\operatorname{Re}\lambda > 0$, then the improper integral $\int_0^\infty e^{-\lambda t}T(t)xdt$ exists for all $x \in X$; necessarily, every such λ belongs to $\rho(A)$ and the formula $R(\lambda, A)x = \int_0^\infty e^{-\lambda t}T(t)xdt$ holds.

By our hypothesis (1.2), we can choose a $z_0 \in \mathbb{C}$ such that $\beta = \operatorname{Re} z_0 > 0$ and

$$M_{z_0} := \beta - (2M||\exp_{-\beta}||_{E^\star})^{-1} < 0.$$

Then for every $\lambda \in \mathbb{C}$ with $M_{z_0} < \operatorname{Re}\lambda < 0$ the point $\lambda_0 = \operatorname{Re} z_0 + i \operatorname{Im}\lambda$ belongs to $\rho(A)$. Since

$$\begin{aligned} |\lambda - \lambda_0| &= \operatorname{Re}\lambda_0 - \operatorname{Re}\lambda < (2M||\exp_{-\beta}||_{E^\star})^{-1} \\ &\le \frac{1}{2||R(\lambda_0, A)||} < \frac{1}{||R(\lambda_0, A)||}, \end{aligned}$$

this yields that λ also belongs to $\rho(A)$ and

$$||R(\lambda, A)|| \le \frac{||R(\lambda_0, A)||}{1 - |\lambda - \lambda_0|||R(\lambda_0, A)||} \le 2M||\exp_{-\beta}||_{E^\star}.$$

Finally, the Gearhart-Prüss Theorem (see [2], [10]) allows us to conclude that $\omega_0(\mathbf{T}) \le M_{z_0} < 0$. □

3. Applications

In this section we consider a linear operator $A : D(A) \subset H \to H$ acting on the complex Hilbert space H, that generates a strongly continuous semigroup $\mathbf{T} = \{T(t)\}_{t\ge 0}$.

Theorem 2. *Under the above assumptions on* A, *if moreover*

(*i*) Φ *verifies the* Δ_2*-condition and* $M = \sup_{||x||\le 1} \rho^\Phi(|\langle T(\cdot)x, x\rangle|) < \infty$;

(ii) the corresponding dual function Φ^* is strictly increasing on $[0,\infty)$ and

$$\liminf_{\alpha\downarrow 0}[\alpha||\exp_{-\alpha}||_{L^{\Phi^*}}]=0,$$

then for each $b\in H$ and each $u^*(\cdot)$ in L^{Φ^*}, the following infinite time Cauchy Problem

$$(A,b,-\infty,0):\quad\left\{\begin{array}{rcl}\dot{x}(t)&=&Ax(t)+bu^*(-t)\quad \text{for } t\le 0\\ x(-\infty)&=&\lim\limits_{t\to-\infty}x(t)=0,\end{array}\right.$$

has a unique solution on $(-\infty,0]$.

Proof. First we shall prove that the function ϕ given by the improper integral

$$\phi(t)=\int_{-\infty}^{t}T(t-\tau)u^*(-\tau)bd\tau=\lim_{s\to-\infty}\int_s^t T(t-\tau)u^*(-\tau)bd\tau,$$

is correctly defined on $(-\infty,0]$. In fact, using the Hölder inequality, for all $t_1<t_2$ in $(-\infty,t]$, we get

$$\begin{aligned}\left\|\int_{t_1}^{t_2}T(t-\tau)u^*(-\tau)bd\tau\right\|&\le\sup_{||y||\le 1}\int_{t_1}^{t_2}|\langle T(t-\tau)b,y\rangle|\cdot|u^*(-\tau)|d\tau\\ &\le\sup_{||y||\le 1}\int_{t-t_2}^{t-t_1}|\langle T(\rho)b,y\rangle|\cdot|u^*(\rho-t)|d\rho\\ &\le\sup_{||y||\le 1}\int_0^{\infty}1_{[t-t_2,t-t_1]}(\rho)|\langle T(\rho)b,y\rangle|\cdot|u^*(\rho-t)|d\rho\\ &\le 2M||b||\rho^{\Phi^*}(1_{[t-t_2,t-t_1]}(\cdot)|u^*(\cdot-t)|).\end{aligned}$$

Taking into account that L^{Φ^*} is rearrangement invariant, we have the relations

$$\begin{aligned}\rho^{\Phi^*}(1_{[t-t_2,t-t_1]}(\cdot)|u^*(\cdot-t)|)&=\rho^{\Phi^*}(1_{[-t_2,-t_1]}(t+\cdot)|u^*(\cdot)|)\\ &=\rho^{\Phi^*}(1_{[-t-t_2,-t-t_1]}(\cdot)|u^*(\cdot)|).\end{aligned}$$

Put $s_1=-t-t_2$ and $s_2=-t-t_1$. Then $0\le -t\le s_1<s_2<\infty$ and, conversely, all such pairs s_1,s_2 come this way.

Given $0<\eta\le 1$, the function $\frac{1}{\eta}u^*(\cdot)$ belongs to L^{Φ^*}, which yields $\int_0^\infty\Phi^*(\frac{1}{\eta}|u^*(\tau)|)d\tau<\infty$. Therefore there exists $\delta>0$ such that for all $\delta\le s_1<s_2<\infty$ we have

$$\int_{s_1}^{s_2}\Phi^*\left(\frac{1}{\eta}|u^*(\tau)|\right)d\tau=\int_0^\infty\Phi^*\left(1_{[s_1,s_2]}(\tau)\frac{1}{\eta}|u^*(\tau)|\right)d\tau\le\eta\le 1.$$

This gives us

$$\rho^{\Phi^*}(1_{[-t-t_2,-t-t_1]}(\cdot)|u^*(\cdot)|)\le\eta,$$

whenever $t_1 < t_2 < -\delta$. In fact,

$$\eta \in \{k > 0 : \int_{s_1}^{s_2} \Phi^* \left(\frac{1}{k} |u^*(\tau)| \right) d\tau = \int_0^\infty \Phi^* \left(1_{[s_1,s_2]} \frac{1}{k} |u^*(\tau)| \right) d\tau \leq 1 \}.$$

Clearly, ϕ verifies the integral equation

$$x(t) = T(t-s)x(s) + \int_s^t T(t-\tau)u^*(-\tau)bd\tau, \quad s \leq t \leq 0.$$

Moreover, for each $t < 0$ we have

$$\begin{aligned}
\|\phi(t)\| &= \left\| \int_{-\infty}^t T(t-\tau)u^*(-\tau)bd\tau \right\| \\
&= \sup_{||y|| \leq 1} \int_0^\infty |\langle T(\rho)b, y \rangle| \cdot |u^*(\rho - t)| d\rho \\
&\leq 2M||b||\rho^{\Phi^*}(|u^*(\cdot - t)|) = \rho^{\Phi^*}(1_{[-t,\infty)}(\cdot)|u^*(\cdot)|).
\end{aligned}$$

On the other hand $\rho^{\Phi^*}(1_{[-t,\infty)}(\cdot)|u^*(\cdot)|) \to 0$ as $t \to -\infty$. Indeed, for $1 \geq \varepsilon > 0$ arbitrarily fixed and $t < 0$ sufficiently small, we have

$$\int_0^\infty \Phi^* \left(1_{[-t,\infty)}(s) \frac{|u^*(s)|}{\varepsilon} \right) ds = \int_{-t}^\infty \Phi^* \left(\frac{1}{\varepsilon} |u^*(s)| \right) ds < \varepsilon.$$

Then $\lim_{t \to -\infty} \phi(t) = 0$, which ends the proof of the fact that ϕ is a solution of the problem $(A, b, -\infty, 0)$. □

Theorem 3. *Assume that Φ satisfies the condition (1.2). If for each $b \in H$ and each $u^*(\cdot) \in (L^\Phi)^*$ the infinite time Cauchy Problem $(A, b, -\infty, 0)$ has a unique solution, then the semigroup generated by A is uniformly exponentially stable.*

Proof. Let E the set of all H-valued bounded and continuous functions g defined on $(-\infty, 0]$. Endowed with the norm $|g|_E := \sup_{t \leq 0} |g(t)|$, the set E becomes a Banach space. Let $b \in H$ and $h > 0$ be fixed and denote by x_{u^*} the unique solution of $(A, b, -\infty, 0)$. We will consider the bounded linear operators P and Q, defined by:

$$u^* \mapsto Qu^* := x_{u^*} : (L^\Phi)^* \to E \text{ and } g \mapsto Pg := g(0) : E \to H.$$

Since PQ is bounded we infer the existence of a positive constant K_b such that

$$|| \int_{-\infty}^0 T(-\tau)u^*(-\tau)d\tau || \leq K_b ||u^*||_{(L^\Phi)^*} \quad \text{for all } u^* \in (L^\Phi)^*.$$

Then for each $u^* \in (L^\Phi)^*$ with $u^*(s) = 0$ for all $s > h$, we have that

$$|\int_0^T \langle T(\tau)b, y\rangle u^*(\tau)d\tau| \leq K_b||u^*||_{(L^\Phi)^*} \text{ for all } y \in H,\ ||y|| \leq 1,$$

and because $(L^\Phi)^*$ is a Banach function space, the previous inequality actually works for all $u^* \in (L^\Phi)^*$. Equivalently,

$$|\int_0^\infty 1_{[0,h]}(\tau)\langle T(\tau)b, y\rangle u^*(\tau)d\tau| \leq K_b||u^*||_{(L^\Phi)^*},$$

for all $y \in H$, $||y|| \leq 1$, and all $u^* \in (L^\Phi)^*$. Now it is easy to see that

$$\rho^\Phi(1_{[0,h]}(\cdot)|\langle T(\cdot)b, y\rangle|) \leq K_b \quad \text{for all } y \in H,\ ||y|| \leq 1.$$

Therefore

$$\rho^\Phi(|\langle T(\cdot)b, y\rangle|) \leq K_b \quad \text{for all } y \in H, ||y|| \leq 1,$$

and from Theorem 1 we can conclude that the semigroup $\mathbf{T}$ is uniformly exponentially stable. □

Assume that for each $x, y \in H$ the map $\langle T(\cdot)x, y\rangle$ defines an element of L^Φ. Then the map given by the formula

$$(x, y) \mapsto \langle T(\cdot)x, y\rangle : H \times H \to L^\Phi$$

is a continuous sesquilinear function (linear in the first variable and anti-linear in the second one). By the Closed Graph Theorem we get the existence of a positive constant M such that

$$\rho^\Phi(|\langle T(\cdot)x, y\rangle|) \leq M||x|| \cdot ||y|| \quad \text{for all } x, y \in H.$$

This shows that the condition (1.1) can be replaced by the following one,

$$\int_0^\infty \Phi(|\langle T(t)x, y\rangle|)dt < \infty, \quad \text{for all } x, y \in H. \tag{3.1}$$

We conclude our paper with an example.

Let $H = L^2[0, \pi]$ and $A : D(A) \subset H \to H$ given by $Ax = \frac{d^2x}{d\xi^2}$, where the domain $D(A)$ consists of all absolutely continuous functions $x(\cdot)$ defined on $[0, \pi]$, which verify the following three conditions: $i)$ $x(0) = x(\pi) = 0$; $ii)$ the first derivative $\frac{dx}{d\xi}$ is absolutely continuous on $[0, \pi]$; $iii)$ the second derivative $\frac{d^2x}{d\xi^2}$ belongs to H. With the above notations, for each $u^*(\cdot) \in (L^\Phi)^*$ and each $b(\cdot) \in H$, the infinite time

Cauchy Problem

$$\frac{\partial y(t,\xi)}{\partial t} = \frac{\partial^2 y(t,\xi)}{\partial \xi^2} + u^*(-t)b(\xi) \quad \text{for } t \in (-\infty, 0],\ \xi \in (0,\pi)$$

$$\lim_{t\to-\infty} \int_0^\pi |y(t,\xi)|^2 d\xi = 0$$

has a unique solution. Indeed, the uniform growth bound $\omega_0(\mathbf{T})$ of the semigroup $\mathbf{T}$ generated by A is equal to -1 and condition (3.1) applies (due to the fact that Φ is a convex function).

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